HERMITE-HADAMARD TYPE INEQUALITY FOR LOG-CONVEX FUNCTIONS VIA SUGENO INTEGRALS

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Abstract. In this paper, Hermite-Hadamard type inequality for Sugeno integrals based on log-convex functions is studied. Some examples are given to illustrate the results.

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [1]. The properties and applications of Sugeno-integral have been studied by lots of authors. Between these others, Ralescu and Adams [2] proposed several equivalent definitions of fuzzy integrals; Román-Flores et al. [3, 4] defined the level-continuity of fuzzy integrals and the H-continuity of fuzzy measures; the book by Wang and Klir [5] contains a general overview on fuzzy measurement and fuzzy integration theory.

Many authors generalized the Sugeno integral by using some other operators to replace the special operators $\vee$ and/or $\wedge$. Suárez García and Gil Álvarez [6] presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals.

In recent years, some authors [7]-[11] generalized several classical integral inequalities for fuzzy integral.Caballero and Sadarangani [11] showed off a Hermite-Hadamard type inequality of fuzzy integrals for convex function. Li, Song and Yue [12] served Hermite-Hadamard type inequality for Sugeno integrals. In [13], Dragomir and Mond introduced to Hermite-Hadamard type inequality for log-convex functions.

The aim of this paper is to prove a Hermite-Hadamard type inequality for Sugeno integrals related to log-convex functions. Some example are given to illustrate the results.

Let's see some proporties of fuzzy integral.

2. Preliminary Discussions

In this section, we remember some basic definition and properties of fuzzy integral and log-convex function. For details we refer the readers to Refs [1, 5, 12].

Suppose that Σ is a σ-algebra of subsets of X and that $\mu : \Sigma \to [0, \infty)$ is a non-negative, extended real-valued set function. We say that μ is a fuzzy measure if and only if:

(1) $\mu(\emptyset) = 0$;

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(2) $E, F \in \Sigma$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);

(3) $\{E_n\} \subset \Sigma, E_1 \subset E_2 \subset ...,$ imply $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$ (continuity from below);

(4) $\{E_n\} \subset \Sigma, E_1 \supset E_2 \supset ..., \mu(E_1) < \infty,$ imply $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$ (continuity from above).

If f is a non-negative real-valued function defined on X, we denote the set

$$\{x \in X : f(x) \geq \alpha\} = \{f \geq \alpha\}$$

by F_α for $\alpha \geq 0$. Note that if $\alpha \leq \beta$ then $F_\beta \subset F_\alpha$.

Let (X, Σ, μ) be a fuzzy measure space, we denote M^+ the set of all non-negative measurable functions with respect to Σ

Definition 2.1. [1, 5] *Let $A \in X$, $f \in M^+$ The fuzzy integral of f on A with respect to μ which is denoted by $(s)\int f d\mu$, is defined by*

$$(s)\int f d\mu = \bigvee_{\alpha \geq 0} \left[\alpha \wedge \mu(A \cap \{f \geq \alpha\})\right].$$

When $A = \Sigma$, the fuzzy integral may also be denoted by $(s)\int f d\mu$.
Where $\vee$ and $\wedge$ denote the operations inf and sup on $[0, \infty)$, respectively.

The following properties of the Sugeno integral are well known and can be found in.

Proposition 2.1. *Let (X, Σ, μ) be a fuzzy measure space, $A \in \Sigma$ and $f, g \in M^+$*

(1) $(s)\int_A f d\mu \leq \mu(A)$;

(2) $(s)\int_A k d\mu = k \wedge \mu(A)$, *$k$ non-negative constant;*

(3) *If $f \leq g$ on A then* $(s)\int_A f d\mu \leq (s)\int_A g d\mu$;

(4) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (s)\int_A f d\mu \geq \alpha$;

(5) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (s)\int_A f d\mu \leq \alpha$;

(6) $(s)\int_A f d\mu < \alpha \Leftrightarrow$ *there exists $\gamma < \alpha$ such that* $\mu(A \cap \{f \geq \gamma\}) < \alpha$;

(7) $(s)\int_A f d\mu > \alpha \Leftrightarrow$ *there exists $\gamma > \alpha$ such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.1. *Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then, due to (4) and (5) of Preposition 2.1, we have that*

$$F(\alpha) = \alpha \Rightarrow (s)\int f d\mu = \alpha.$$

Thus, from a numerical point of view, the fuzzy integral can be calculated solving the equation $F(\alpha) = \alpha$.

[14], J.Caballero, K. Sadarangani proved with the help of certain examples that the classical Hermite-Hadamard inequalities for fuzzy integrals.

Definition 2.2. [13] *Let I be an interval of real numbers. A function $f : I \to (0, \infty)$ is said to be log-convex or multiplicaticely convex if $\log(f)$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:*

$$f(tx + (1-t)y) \leq |f(x)|^t |f(y)|^{1-t}.$$

We note that if f and g are convex functions and g is monotonic nondecrasing, then $g \circ f$ is convex. Moreover, since $f = \exp(\log(f))$, it follows that a log-convex function is convex, but the converse is not true.

3. Hermite-Hadamard Type Inequality for Preinvex Functions via Sugeno Integrals

The following inequality is well known in the literature as the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$.

In [13], S.S. Dragomir extended this classic result for log-convex functions as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq L(f(a), f(b)),$$

where $L(p,q) := \dfrac{p-q}{\ln p - \ln q}$ $(p \neq q)$ is the logaritmic mean of the positive real numbers p, q (for $p = q$, we put $L(p,p) = p$).

In this paper, we prove using Sugeno integral another refinement of the Hermite-Hadamard type inequality for log-convex functions. Some applications for special means are also given.

Example 3.1. Consider $X = [0, 1]$ and let μ be the Lebesgue measure on X. If we take the function $f(x) = e^{x+1}$, then $f(x)$ is a log-convex function. To calculate the Sugeno integral related to this function, let's consider the distribution function F associated to f to $[0, 1]$, by Remark 2.1, this is

$$\begin{aligned} F(\alpha) &= \mu([0,1] \cap \{f \geq \alpha\}) = \mu\left([0,1] \cap \left\{e^{-x} \geq \alpha\right\}\right) \\ &= \mu([0,1] \cap \{x \leq -\ln(\alpha)\}) = -\ln(\alpha). \end{aligned}$$

and we solve the equation

$$-\ln(\alpha) = \alpha.$$

It is easily proved that the solutions of the last equation is 0.5672 with using bisection method of numerical analysis, and, Remark 2.1, we get

$$(s)\int_0^1 f d\mu = (s)\int_0^1 e^{x+1} d\mu = 0.5672.$$

On the other hand,

$$f\left(\frac{0+1}{2}\right) = f\left(\frac{1}{2}\right) = e^{-\frac{1}{2}} = 0.6065.$$

This proves that the left part of Hermite-Hadamard inequality is not satisfied in the fuzzy context.

Example 3.2. Consider $X = [0,1]$ and let μ be the Lebesque measure on X. Then for the log-convex function $f(x) = e^{-\cos(x)-1}$ and using a similar argument that in Example 3.1, we can get

$$(s)\int_0^1 f d\mu = (s)\int_0^1 \left(e^{-\cos(x)-1}\right) d\mu = 0.1852$$

On the other hand,

$$L\left(f(0), f(1)\right) = \frac{f(0) - f(1)}{\ln f(0) - \ln f(1)} = 0.1718$$

and this proves that right-hand side of Hermite-Hadamard inequality is not satisfied for fuzzy integrals.

The aim of the following theorem is to show a Hermite-Hadamard type inequality for the Sugeno integral.

Theorem 3.1. *Let $g : [0,1] \to [0,\infty)$ be a log-covex function such that $g(0) < g(1)$ and μ the Lebesque measure on $\mathbb{R}$. Then*

$$(s)\int_0^1 g d\mu \leq \min\{\alpha, 1\},$$

where $\alpha = 1 - t$, t satisfies the following equation

$$[g(0)]^{1-t} . [g(1)]^{t} + t - 1 = 0$$

Proof. As a g is a log-convex function, for $x \in [0,1]$

$$g(x) = g((1-x).0 + x.1) \leq [g(0)]^{1-x} . [g(1)]^{x} = h(x)$$

hence, by (3) of Proposition 2.1, we have that

$$\begin{aligned} (s)\int_0^1 g d\mu &\leq (s)\int_0^1 g((1-x).0 + x.1) d\mu \\ &\leq (s)\int_0^1 [g(0)]^{1-x} . [g(1)]^{x} d\mu = (s)\int_0^1 h(x) d\mu. \end{aligned}$$

In order to calculate the integral in the right-hand part of the last inequality, we consider the distribution function F given by

$$\begin{aligned} F(\alpha) &= \mu([0,1]\cap\{h\geq\alpha\}) = \mu\left([0,1]\cap\left\{[g(0)]^{1-x}\cdot[g(1)]^{x}\geq\alpha\right\}\right) \\ &= \mu\left([0,1]\cap\left\{x\geq\frac{\ln(\alpha)-\ln(g(0))}{\ln(g(1))-\ln(g(0))}\right\}\right) \\ &= 1-\frac{\ln(\alpha)-\ln(g(0))}{\ln(g(1))-\ln(g(0))}, \end{aligned}$$

and the solution of the equation

$$1-\frac{\ln(\alpha)-\ln(g(0))}{\ln(g(1))-\ln(g(0))}=\alpha.$$

Let $\alpha=1-t$, t satisfies the following equation

$$[g(0)]^{1-t}\cdot[g(1)]^{t}+t-1=0.$$

By (1) of Proposition 2.1, we get that

$$(s)\int_0^1 h(x)\,d\mu\leq\mu([0,1])=1.$$

By Remark 2.1, we have

$$(s)\int_0^1 gd\mu\leq\min\{\alpha,1\}.$$

This completes is proof.

Remark 3.1. *In the case $g(0)=g(1)$ in Theorem 3.1, the function $h(x)$ is*

$$h(x)=[g(0)]^{1-x}\cdot[g(1)]^{x}=g(0)$$

and

$$(s)\int_0^1 gd\mu\leq(s)\int_0^1 h(x)\,d\mu=(s)\int_0^1 g(0)\,d\mu=g(0)\wedge 1.$$

Theorem 3.2. *Let $g:[0,1]\to[0,\infty)$ be a log-convex function such that $g(0)>g(1)$ and μ the Lebesque measure on $\mathbb{R}$. Then*

$$(s)\int_0^1 gd\mu\leq\min\{\alpha,1\}$$

where α is root of the equation

$$\frac{\ln(\alpha)-\ln(g(0))}{\ln(g(1))-\ln(g(0))}=\alpha.$$

Let $\alpha=1-t$, t satisfies the following equation

$$[g(0)]^{t}\cdot[g(1)]^{1-t}+t-1=0.$$

Proof. Similarly, using the method in Theorem 3.1, we have

$$\begin{aligned} F(\alpha) &= \mu([0,1] \cap \{g \geq \alpha\}) \\ &= \mu\left([0,1] \cap \left\{x \leq \frac{\ln(\alpha) - \ln(g(0))}{\ln(g(1)) - \ln(g(0))}\right\}\right) \\ &= \frac{\ln(\alpha) - \ln(g(0))}{\ln(g(1)) - \ln(g(0))}, \end{aligned}$$

and the solution of the equation

$$\frac{\ln(\alpha) - \ln(g(0))}{\ln(g(1)) - \ln(g(0))} = \alpha,$$

where α satisfies the following equation

$$[g(0)]^{1-\alpha} . [g(1)]^{\alpha} - \alpha = 0.$$

The proof of the rest part is similar, so we omit it.

Example 3.3. Consider $f(x) = e^{x^2-1}$ on $[0,1]$. Obviously, this function is non-negative, non-decreasing and log-convex on the interval $[0,1]$. Moreover, $f(0) = e^{-1} = \frac{1}{e}$ and $f(1) = 1 > 0$. Calculating the fuzzy integral, we have

$$1 - \frac{\ln(\alpha) - \ln(f(0))}{\ln(f(1)) - \ln(f(0))} = \alpha.$$

Then, solving by bisection method of numerical analysis, the approximately solution $\alpha = 0.5672$. By Theorem 3.1, we have

$$(s)\int_0^1 f d\mu \leq \min\{\alpha, 1\} = 0.5672.$$

Also t is the root of the $\alpha = 1 - t$ equation, satisfies the following equation

$$e^{t-1} + t - 1 = 0.$$

Example 3.4. Consider the log-convex function $f(x) = e^{-\sin(x)}$, for $x \in [0,1]$. Then $f(0) = 1$ and $f(1) = 0.4311$, and we have

$$\frac{\ln(\alpha) - \ln(f(0))}{\ln(f(1)) - \ln(f(0))} = \alpha$$

which gives by solving by bisection method of numerical analysis, the approximately solution $\alpha = 0.6024$, satisfies under the equation

$$\ln(\alpha) + \sin(1) * \alpha = 0.$$

By Theorem 3.2, we have estimate:

$$(s)\int_0^1 e^{-\sin(x)} d\mu \leq \min\{\alpha, 1\} = \alpha.$$

Theorem 3.3. *Let $g : [a,b] \to [0,\infty)$ be a log-convex function and μ the Lebesque measure on $\mathbb{R}$. Then*

(i) If $g(a) < g(b)$, then

$$(s)\int_a^b g d\mu \leq \min\{\alpha_1, b-a\}$$

where α_1 is root of the equation

$$b - \frac{(b-a).\ln(\alpha) - b.\ln(g(a)) + a.\ln(g(b))}{\ln(g(b)) - \ln(g(a))} = \alpha.$$

(ii) If $g(a) = g(b)$, then

$$(s)\int_a^b g d\mu \leq \min\{g(a), b-a\}.$$

(iii) If $g(a) > g(b)$, then

$$(s)\int_a^b g d\mu \leq \min\{\alpha_2, b-a\},$$

where α_2 is root of the equation

$$\frac{(b-a).\ln(\alpha) - b.\ln(g(a)) + a.\ln(g(b))}{\ln(g(b)) - \ln(g(a))} - a = \alpha.$$

Proof. We will prove (i) and other two cases are similar. Note that as g is a log-convex function then for $x \in [0,1]$ we have

$$g(x) = g\left(\left(1 - \frac{x-a}{b-a}\right).a + \frac{x-a}{b-a}.b\right) \leq (g(a))^{\frac{b-x}{b-a}}.(g(b))^{\frac{x-a}{b-a}} = h(x).$$

By (3) of Proposition 2.1,

$$(s)\int_a^b g d\mu \leq (s)\int_a^b (g(a))^{\frac{b-x}{b-a}}.(g(b))^{\frac{x-a}{b-a}} d\mu = (s)\int_a^b h(x) d\mu.$$

Now, we consider the distribution function F given by

$$\begin{aligned}
F(\alpha) &= \mu([a,b] \cap \{h \geq \alpha\}) = \mu\left([a,b] \cap \left\{(g(a))^{\frac{b-x}{b-a}}.(g(b))^{\frac{x-a}{b-a}} \geq \alpha\right\}\right) \\
&= \mu\left([a,b] \cap \left\{x \geq \frac{(b-a).\ln(\alpha) - b.\ln(g(a)) + a.\ln(g(b))}{\ln(g(b)) - \ln(g(a))}\right\}\right) \\
&= b - \frac{(b-a).\ln(\alpha) - b.\ln(g(a)) + a.\ln(g(b))}{\ln(g(b)) - \ln(g(a))},
\end{aligned}$$

and the root is α_1 which is the solution of the equation

$$b - \frac{(b-a).\ln(\alpha) - b.\ln(g(a)) + a.\ln(g(b))}{\ln(g(b)) - \ln(g(a))} = \alpha.$$

Then by (1) of Proposition 2.1 and Remark 2.1, we have

$$(s)\int_a^b g d\mu \leq (s)\int_a^b h(x)\, d\mu = \min\{\alpha_1, b-a\}.$$

Example 3.5. Consider $f(x) = e^{-\sin(2x)}$ be a function defined on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. This function is non-decreasing and log-convex because $\log(\exp(-sin(2x))) = -\sin(2x)$ function is convex and $f(x) = e^{-\sin(2x)}$ is non-negative. As $f\left(\frac{\pi}{4}\right) = 0.3679$ and $f\left(\frac{\pi}{2}\right) = 1$ and $f\left(\frac{\pi}{4}\right) < f\left(\frac{\pi}{2}\right)$, by (a) of Theorem 3.3 we can get the following estimate:

$$(s)\int_{\pi/4}^{\pi/2} \left(e^{-\sin(2x)}\right) d\mu \leq \min\left\{\alpha_1, \frac{\pi}{4}\right\}$$

where α_1 is root which is the equation

$$\frac{\pi}{4} - \frac{\left(\frac{\pi}{4}\right).\ln(\alpha) - \frac{\pi}{2}.\ln\left(g\left(\frac{\pi}{4}\right)\right) + \frac{\pi}{4}.\ln\left(g\left(\frac{\pi}{2}\right)\right)}{\ln\left(g\left(\frac{\pi}{2}\right)\right) - \ln\left(g\left(\frac{\pi}{4}\right)\right)} = \alpha.$$

This equation have been solved by matlab program and the root is $\alpha_1 = 0.5175$. Definitively Sugeno integral:

$$(s)\int_{\pi/4}^{\pi/4} \left(e^{-\sin(2x)}\right) d\mu \leq \min\left\{\alpha_1, \frac{\pi}{4}\right\} = \alpha_1 = 0.5175.$$

4. Conclusion

In this paper, we have researched the classical Hermite-Hadamard inequality for Sugeno integral based on log-convex function. For further investigations we will continue to study Hermite-Hadamard and other integral inequalities for several fuzzy integrals based on log-convex function.

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